5. ALL ABOUT ALICE

Students will:

- extend the concept of exponent to include zero, negative, and fractional exponents
- develop and use the Additive Law of Exponents, $a^b \times a^c = a^{b+c}$, and other general properties of exponents
- develop and analyze the graph of the function $y = 2^x$ and related functions
- define and work with logarithms, both on paper and using calculators
- write numbers using scientific notation and use this notation to simplify computation
- work with scientific notation on calculators
- learn about order of magnitude, and use this concept in estimation
- understand the concept of significant digits, and use this concept in context
- work with different units of measurement, and convert from one to another
- be introduced to and use standard f(x) function notation
- study and and use basic ideas of logical inference
- work with mirror symmetry

Integrated Math 3 Content

The course is divided into 5 units. The following is a summary of the concepts and skills students will encounter and practice in each unit.

1. FIREWORKS

Students will:

- use formulas in solving problems
- learn terminology related to quadratic expressions, functions, and equations
- understand the role of the vertex and the x-intercept in the graphs of quadratic functions
- recognize the significance of the sign of the X^2
- transform quadratic expressions into vertex form
- apply graphing to solving problems involving quadratics
- use the zero property of multiplication to solve quadratics
- use factoring to solve quadratics
- relate factoring of quadratics to problems of area
- understand meaning of "not factorable"
- identify certain expressions as perfect squares

2. ORCHARD HIDEOUT

Students will:

- use the Cartesian coordinate system to organize a complex problem
- review the concepts of area and the Pythagorean Theorem
- find the equation for a circle with an arbitrary center and radius
- understand the use of the phrases "if . . ., then. .." and "if and only if" in definitions and proofs
- define the concept of congruence and examine its relationship to similarity
- discover and prove that the set of points equidistant from two given points is the perpendicular bisector
- prove that the perpendicular bisectors of a triangle must meet in a common point
- find criteria which will guarantee the triangles are congruent
- find the relationship of the circumference and area of a circle to its radius
- discover that the ratios of areas and circumferences of circles to areas and perimeters of circumscribed regular polygons are independent of the radius of the circles
- develop formulas for the area and perimeter of polygons circumscribed about a circle, and use these to develop and understand formulas for the area and circumference of a circle
- learn about the significance of pi
- discover that any line through the midpoint of a segment is equidistant from the endpoints
- discover and prove that the set of points equidistant from two intersecting lines is the union of the bisectors of the angles formed by the lines
- prove that the three angle bisectors of a triangle must meet in a common point
- find ruler and straightedge constructions for various geometric configurations
- develop and apply the distance and midpoint formulas
- practice spatial visualization
- use a compass to create art

3. Meadows or Malls?

Students will:

- Express constraints stated in words as inequalities using algebra
- Find numbers that satisfy constraints
- Review linear programming with two variables
- Solve systems of more than two equations by substitution
- Solve linear systems of equations by elimination
- Geometrically represent systems of linear equations in two and three variables
- Study the possible ways in which sets of lines and planes intersect
- Use algebra to write a proof
- Develop definitions for addition and multiplication of matrices
- Express complex manipulations in words
- Learn which algebraic laws hold for operations on the real numbers
- Learn which algebraic laws hold for matrix operations
- Learn that matrix equations are equivalent to systems of linear equations
- Understand the use of matrix algebra to solve systems of linear equations
- Find matrix inverses on paper and with a TI-82/83
- Understand the relationship of inconsistent and dependent equations to non invertible matrices
- Use the TI-82/83 and inverses of matrices to solve systems of linear equations
- Present the analysis and solution of complex linear programming problems

4. SMALL WORLD

Students will:

- analyze rates of change from graphs
- realize that, other things being equal, the rate of change of population growth is proportional to the population
- discover and prove that lines have a constant slope and rate of change
- understand the relationship between the rate of change of a function and the appearance of its graph
- develop an algebraic definition of slope
- understand the significance of a negative slope for the appearance of a graph
- develop the concept of the derivative of a function at a point as a generalization of the notion of slope
- numerically estimate derivatives of curves at points
- strengthen understanding of logarithms
- discover and explain the properties and relationships of logarithms and exponents
- realize the derivative of exponential functions are proportional to their values
- characterize situations appropriately with exponential functions
- develop recursive rules
- estimate the value for e as the base of the exponential function whose derivative equals its value
- develop and use a formula for compound interest
- learn the meaning of the terms natural and common logarithms
- use exponential functions to fit data
- explore patterns and find formulas concerning arithmetic sequences and their partial sums

5. Pennant Fever

Concepts and Skills

Here is a summary of the main concepts and skills that students will encounter and practice in this unit.

Probability and statistics

- Developing a mathematical model for a complex probability situation
- Using area diagrams and tree diagrams to find and explain probabilities
- Using a simulation to understand a situation, to help analyze probabilities, and to support a theoretical analysis
- Finding expected value
- Finding and using probabilities for sequences of events
- Using specific problem contexts to develop the binomial distribution, and finding a formula for the associated probabilities
- Using probability to evaluate null hypotheses

Counting principles

- Developing systematic lists for complex situations
- Using the multiplication principle for choosing one element from each of several sets
- Defining and using the concepts of permutation and combination
- Understanding and using standard notation for counting permutations and combinations
- Developing formulas for the permutation and combinatorial coefficients

Pascal's triangle and combinatorial coefficients

- Finding patterns and properties within Pascal's triangle
- Recognizing that Pascal's triangle consists of combinatorial coefficients
- Explaining the defining pattern and other properties of Pascal's triangle using the meaning of combinatorial coefficients
- Developing and explaining the binomial theorem

Other concepts and skills are developed in connection with Problems of the Week.

Integrated Math 4 Content

The course is divided into 5 units. The following is a summary of the concepts and skills students will encounter and practice in each unit.

1. High Dive

Students will:

- Review basics about circles, angles, and right triangle trigonometry,
- Review the concept of periodicity,
- Examine a complex problem to determine what information is needed to solve it,
- Find a quadrant-by-quadrant expression for circular motion.
- Extend the sine and cosine functions to all angles, based on both phyusical models and coordinate reasoning,
- See the importance of similarity in the definition of the sine and cosine functions,
- Connect the sine and cosine functions to Ferris wheel problems.
- Graph the sine and cosine functions,
- Define the inverse sine function and principal values,
- Work with varying parameters to model examples of periodicity,
- Review the concept of instantaneous velocity and its estimation,
- Review the derivative and its relationship to instantaneous velocity,
- Examine the relationship between average to instantaneous velocity,
- Understand why, for situations with constant acceleration, the average velocity for any interval is equal to the average of the instantaneous velocities of the two endpoints of the interval,
- Develop quadratic expressions for the height of free-falling objects, based on the physical principle of constant acceleration,
- Study the relationship among acceleration, velocity, and speed, with special attention to the signs of acceleration and velocity,
- See the importance of quadratic equations in studying falling objects
- learn how to use the quadratic formula to solve quadratic equations
- discover and explaintrigonometric identities
- define polar coordinates and develop expressions for rectangular coordinates in terms of polar coordinates
- express velocity in terms of its vertical and horizontal components
- study the motion of falling objects and find a general solution for the falling time of objects with an initial vertical velocity

2. The World of Functions

Students will:

- review different ways to think about functions
- formally define functions as sets of ordered pairs
- sketch graphs of functions based on situations
- review basic families of functions
- find, describe, and prove patterns in the tables of functions based on the algebraic form
- study the concept of direct and inverse proportionality and their constants
- find vertical and horizontal asymptotesfor specific functions and vice versa
- find patterns in the graphs of power functions
- compare models of linear and exponential growth
- develop a measure of "quality of fit" for a function to a set of data
- compare functions using the least squares approximation
- use calculator regression to find a function that fits the data
- use absolute value and step functions to model problem situations
- describe situations using arithmetic combinations of functions
- formally define arithmetic operations on functions
- relate arithmetic operations on functions to graphs
- use rational functions to model problem situations
- understand situations involving composition of functions
- define composition and composition notation
- understand situations involving inverse functions
- formally define inverse functions
- relate the concept of inverse function to graphs and tables
- find the inverse of specific functions
- work with iteration of functions

3. Know How

Students will:

- use factoring to solve equations
- learn and use radian measure
- prove the quadratic formula
- use the Laws of Sines and Cosines
- perform complex number operations
- study the ellipse

4. As The Cube Turns

Students will:

- learn to use a technical manual
- use loops in programming
- expres geometric transformations in terms of coordinates in two and three dimensions
- review the algebra of matricies
- use matricies to express geometric transformations
- derive the formula for the area of a triangle in terms of the sine of an angle and adjacent sides
- derive formulas for the sine and cosine of the sum of two angles
- understand programs from their listings
- find the projection of a point on a plane from the perspective of a fixed point, and develop an algebraic description of the projection process
- study the effect of change of viewpoint on projections
- use spatial reasoning in two and three dimensions

5. Pollster's Dilemma

Concepts and Skills

Here is a summary of the main concepts and skills that students will encounter and practice in this unit.

General concepts for sampling

- Establishing methods of good polling, including random sampling
- Using sampling from a known population to analyze reliability of samples
- Distinguishing between "sampling with replacement" and "sampling without replacement," and comparing the two methods
- Using the terminology of "true proportion" and "sample proportion"
- Identifying simplifying assumptions in analyzing sampling

Specific results on sampling with replacement

- Making probability bar graphs for various distributions
- Developing the concept of a theoretical distribution for sampling results from a given population
- Using combinatorial coefficients to find the theoretical distribution of poll results for polls of various sizes
- Generalizing that sampling results fit a binomial distribution

The central limit theorem and the normal distribution

- Seeing intuitively that as poll size increases, the distribution of sample proportions becomes approximately normal
- Reviewing the concept of normal distribution
- Using estimates of areas to understand the normal distribution table
- Applying the central limit theorem for the case of binomial distributions

Mean and standard deviation

- Reviewing the steps for computation of standard deviation
- Seeing that the "large number of trials" method for computing mean and standard deviation is independent of the number of trials
- Extending the concepts of mean and standard deviation from sets of data to probability distributions
- Defining the concept of variance
- Finding formulas for the mean and standard deviation of the distribution of poll results in terms of the poll size and the true proportion
- Deciding what to use for σ if the true proportion is unknown, and finding the maximum value of σ for polling problems

Confidence levels and margin of error

- Using the terminology of "confidence level," "confidence interval," and "margin of error"
- Seeing how poll size affects the standard deviation of poll results
- Establishing confidence intervals in terms of a sample proportion and the standard deviation
- Seeing how the term "margin of error" is commonly used in newspaper reporting
- Estimating the size of a poll based on its reported margin of error

Appendix F: Methodology for Rasch Analysis and Multiple Imputation

This study used a Rasch model (Wright & Stone, 1979) combined with multiple imputation (Rubin, 1987) as an alternate approach for handling missing responses to the Algebra Achievement test. A Rasch model can be thought of as a special case of a class of models called "Item Response Theory" (IRT) models. When implemented together with an IRT model, multiple imputations have often been referred to as "plausible values" (Mislevy, Beaton, Kaplan, & Sheehan, 1992). This appendix discusses the theory behind both multiple imputation/plausible values and Rasch models, and then describes how these concepts were implemented by the current study.

Multiple Imputation

As noted in Chapter 4, there were a number of students who participated in this study and completed 18 of the 19 items on Test 1, as well as a number of students who completed 13 of the 14 items on Test 2. If a student was missing a response to one item on a test, the analysis in Chapter 4 used OLS regression to impute the student's response to that particular item based on her or his responses to the other items on the test. Using this imputed response, the student's mean score on the test was computed and analyzed as part of the complete data set.

Rubin (1987) has demonstrated that this "single imputation" approach to missing data is not ideal because it underestimates the variance that would have been observed had all responses been available. Instead, it is better to use "multiple imputation." Multiple imputation creates the probability distribution of each student's possible responses, and then for each student draws a response (known as a "plausible value") from this probability distribution. OLS regression parameters are computed for this “draw” of imputed plausible values. Then, the process is repeated. A second “draw” of plausible values is taken from the probability distribution, and a second set of parameters is computed. This can be repeated any number of times. The mean value of the parameter estimates from all the draws is the reported parameter estimate, which has a t-distribution with degrees of freedom and standard deviation that can be computed with formulas reported below. Even two “draws” of plausible values provide a relatively efficient parameter estimate and accurate confidence interval. Five draws are commonly used, and in most cases five draws provide efficiency and accuracy of confidence intervals approaching what could be accomplished with an infinite number of draws.

While theoretically superior to single imputation, using multiple imputation for a missing student response to one item would have been rather like swatting a mosquito with a sledge hammer. The missing response comprised, respectively, one-nineteenth of Test 1 or one-fourteenth of Test 2, and underestimating the variance of that one response was unlikely to impact final results. Multiple imputation was unlikely to be familiar to most readers of the current study. Implementing such a complex methodology to account for a missing response to a single item would thus provide little advantage, while potentially confusing readers of the study.

However, students who attended only the first day of testing, while responding to 18 of the 19 items on Test 1, responded to only 1 of the 14 items on Test 2. For this reason, there was very little data available to estimate their Test 2 score, and the Test 2 scores of these students were not used in the primary analysis reported in Chapter 4. Similarly, students who attended only the second day of testing responded to only 1 of the 19 items on Test 1, and their Test 1 scores were not used in the primary analysis reported in Chapter 4.

Dropping these scores was a potential problem, because it seemed reasonable to suppose that students who missed one day of testing might have systematically different achievement than students who attended both days. Doing without these students' data could potentially bias the analysis.

Imputing student scores on Test 1 or Test 2 from responses to a single item is very different from imputing scores from responses to all but one item. Most of the data for these students is missing, and underestimating the variance in the missing data could bias results in important ways. Thus, if the scores of students who responded to a single item were to be included in OLS regression estimates, multiple imputation would be necessary.

Imputing plausible values from student responses to a single item was difficult for two reasons. First, using standard methods like OLS regression for imputing a mean score on a test composed of 14 or 19 items from a score on just one of those items would be likely to provide a very inaccurate estimate. Second, students who were missing data might be systematically different from students with complete data. In the terminology used by Rubin (1987), the non-response may be "nonignorible." (Readers may be more familiar with the concept "missing at random." Rubin has demonstrated that, in most cases, "missing at random" as commonly defined is equivalent to "nonignorable.") In such a situation, even when students with complete data are used to develop a probability distribution of complete scores given the score on just one item, this probability distribution may not be accurate for students with missing data. Such students are perhaps systematically different from the complete-data students whose responses were used to develop the probability distribution. Thus, a method must be devised to modify the probability distribution to account for possible systematic characteristics of students with missing data.

Fortunately, a Rasch model solves both problems. It provides a good way to estimate student abilities based on just one response. Further, it provides a way to modify these estimates to account for systematic differences between responders and non-responders, that is, to adjust for nonignorable non-response. The next section describes how this is accomplished.

The Rasch Model

Arguments have raged over the relative merit of Rasch versus other IRT models. This study chose a Rasch approach over competing IRT models not based upon any theoretical argument, but rather due to the practical considerations that 1) other IRT models often require a larger sample size than was available for this study and 2) an excellent software package implementing a Rasch model was available. This section provides a brief description of the theory behind the Rasch approach.

The simplest Rasch model describes a test composed of dichotomous (right/wrong) items designed to measure a single, unidimensional “ability”. A person’s likelihood of answering any given item correctly is a function of the person’s latent ability β and the item’s difficulty δ. Specifically, the “odds of getting an item correct” is modeled as $e^{\beta-\delta}$, or equivalently, $\beta-\delta$ is the “log-odds” of a person with ability β getting an item with difficulty δ correct.

This study used the ConQuest computer program (Wu, Adams, & Wilson, 1998) to perform its analyses. Like other Rasch modeling software ConQuest simultaneously estimates item difficulties and person abilities, based on person’s responses to a set of items. ConQuest generalizes the Rasch model for dichotomous data in a number of ways. First, it accommodates polytomous responses like those used on the Algebra Achievement test, implementing an adaptation proposed by Masters (1982). Second, the approach of ConQuest is multivariate. For example, in the current analysis ConQuest estimated student scores on Test 1 and Test 2 simultaneously, computing a student’s likely ability on each test while accounting for the correlation between Test 1 and Test 2. Finally, ConQuest uses a Bayesian approach to simultaneously estimate multivariate ability scores and to perform "latent regression", that is, to estimate regression parameters that predict those responses. The approach is iterative: each change in a regression parameter estimate causes changes in estimates of each item's difficulty and of each person's ability, which in turn cause changes in the regression parameter estimates. This process is repeated until the program converges on a solution.

A Rasch model like the one used by ConQuest can handle missing responses more appropriately than other approaches. This is true because the model adjusts a person's ability score based on the difficulty of the items answered. Thus, someone who responds to a few easy items correctly while not having the opportunity to answer the rest of the test will not be assigned as high an ability as someone who responds to a few difficult items correctly while not having an opportunity to answer the rest of the test. Also, the latent regression model used by ConQuest inherently weights a person's ability estimate by the error of measure of that estimate. An ability estimated based upon a few responses will usually not receive as much weight as an ability estimated by many responses, due to the inherent uncertainty in measuring the former.

If non-response were ignorable, ConQuest's iterative approach could provide a valid estimate of regression parameters and standard deviations directly, that is, without needing to run an OLS regression on plausible values or on any other explicit estimate of student ability. See Mislevy (1984) for more details on how this is accomplished. Unfortunately, the non-response in the current study was not ignorable, that is, student responses could not be assumed to be missing at random. For the current study, while some students with missing data scored very high on the part of the test they did complete, on average those with missing data scored noticeably lower on the parts of the test they completed. Thus, it was reasonable to assume that lower-ability students were more likely to be missing responses. For this reason, "plausible values" were utilized for the analysis.

The advantage of having the Rasch model compute plausible values for the current study was as follows. It was possible to estimate plausible values based on all the information in the model, including the partial responses of students with missing information, their Grade 6 ability, and an indicator that they had missed a day of testing. By including the indicator, individuals who missed a day had their estimated "ability" shrunken towards the mean of all students who missed a day. That is, the model accounted for the fact that missing a day predicted lower ability, over and above what was recorded by a person's actual responses. In short, the Rasch model used a different probability distribution for students with missing responses than it used for students with complete responses. It shrunk the estimated score of students who missed a day of testing towards the mean score of all students who missed that day of testing, while shrinking the score of students with complete responses towards the (higher) mean score of all students who attended both days of testing. In this way, the model generated plausible values that accounted for the nonignorable nature of non-response.

Implementation Details

Five draws of plausible values were estimated using as inputs student item responses, sixth grade ability scores, an indicator of whether a student had missed the first day of testing, and an indicator of whether a student had missed the second day of testing. However, the two indicator variables were not utilized in the subsequent five OLS regression runs.

OLS regression parameters estimated from a model including indicators of "absence" would not be appropriate, because they would estimate the effects of Treatment after "controlling for" absence. Because of different testing conditions, students in the Reform cohorts were slightly more likely to have missed a day of testing than were students in the Traditional cohort. "Conditioning out" the absentee variable by including it in the model would bias the results in favor of the Reform cohorts.

However, draws of plausible values estimated while conditioning on the "absentee" variable could legitimately be used in a series of Optimum Least Squares (OLS) regression that did not use the "absentee" variables. This what the current study did.

For each draw of plausible values, OLS regression was used to compute the effect on achievement of being in a Reform cohort, controlling for prior ability but not for the "absentee" indicator variables. For each measure (Test 1 and Test 2) the final estimate of the effect of being in a Reform cohort was computed as the average of the five OLS results. That is, if the output estimate of the Reform cohort effect from the i^{th} draw of plausible values is p_i, then the estimated Reform cohort effect $\bar{Q}$ reported in Chapter 4 is $\bar{Q} = \Sigma p_i/5$. Following the formula of Rubin (1987) variance for $\bar{Q}$ was computed as follows:

Let p_i be a parameter estimate from the i^{th} run of plausible values and σ_i be its standard deviation. Then the estimated variance of $\bar{Q}$ is the sum of two values:

1) $\bar{U}$ = the mean of the five estimates, or mean(σ_i^2), plus
2) (n+1)/n * B where n = number of plausible values drawn, and B= the variance of the n parameter estimates, that is Variance(p_i). In this case, since there were five draws, the B was multiplied by 6/5.

The estimated parameter then has a *t*-distribution, with
Mean = $\bar{Q}$,
Variance = ($\bar{U}$ + (n+1)/n * B), = $\bar{U}$ + 6/5*B
Degrees of Freedom= $(n-1)*(1+r^{-1})^2 = 4*(1+r^{-1})^2$ where
r = (n+1)/n*B/$\bar{U}$ = 6/5*B/$\bar{U}$.

As reported in Chapter 4, results of this analysis led to conclusions little different from the simpler OLS method of dealing with missing data. This confirms that the results reported in Chapter 4 are reasonable, despite the missing data.

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